Use of the FFT to Speed Analysis of Planar Symmetrical 3- and 5-Ports by the Integral Equation Method

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Abstract—**In a recent paper, it was shown that, for planar two-dimensional problems with symmetry, linear eigenvalue-impedance matrix entry relations may be used to simplify the integral equation method of analysis [1]. In this paper, it is pointed out that, in the case of planar circuits with N-fold rotational symmetry, these linear relations take the form of the discrete Fourier transform (DFT). Consequently, the fast Fourier transform (FFT) may be used in its place to give a further substantial improvement in computational speed.**

I. Introduction

With the integral equation method, it is necessary to calculate an $N \times N$ wave impedance matrix $[Z]$ [1]–[3]. The quantity N corresponds to the number of integral subdivisions of the periphery of the planar circuit as in Fig. 1(a) or 1(b). The matrix Z itself is given by the product of two $N \times N$ matrices

$$[Z]=[U]^{-1}\cdot[T] \tag{1}$$

where the entries of the $N \times N$ matrices $[U]$ and $[T]$ are determined by closed-form expressions in terms of known quantities. The majority of the computational effort with this method involves first calculating the inverse of the $N \times N$ matrix U and then calculating the product of the two $N \times N$ matrices $[U]^{-1}$ and $[T]$. For example, to calculate the product of two $N \times N$ matrices, it is required in general to perform N^2 multiplications and $(N-1)^2$ additions. However, with symmetrical networks, the number of independent matrix entries is reduced. This opens up the possibility of reducing the number of basic computations which must be performed. In fact, for the symmetrical networks of Fig. 1(a) or (b) with N-fold rotational symmetry, the matrices $[Z]$, $[U]$, $[T]$, and $[U]^{-1}$ have the following simple form:

$$[U]=\begin{bmatrix} U_1 & U_2 & U_3 & \cdots & U_N \\ U_N & U_1 & U_2 & \cdots & U_{N-1} \\ U_{N-1} & U_N & U_1 & \cdots & U_{N-2} \\ \vdots & & & & \\ U_2 & U_3 & U_4 & \cdots & U_1 \end{bmatrix}. \tag{2}$$

There are only N independent matrix entries U_m. The N-fold rotational symmetry refers only to the circular disk which is divided into N equal segments and not the connecting striplines which are denoted by dashed lines.

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II. Use of the DFT with the Integral Equation Method

In a recent publication, it was shown how the linear eigenvalue-impedance matrix entry relations for N-fold rotationally symmetric circuits can be used to speed up the matrix inversion required by (1) [1]. In particular, the eigenvalues λ_m of the matrix U defined by (2) are given by [1]

$$\lambda_m=\sum_{n=1}^{N} U_n e^{j(m-1)(n-1)2\pi/N}. \tag{3}$$

Similarly, the entries U_m are given in terms of the eigenvalues by

$$U_m=\frac{1}{N}\sum_{n=1}^{N} \lambda_n e^{-j(m-1)(n-1)2\pi/N}. \tag{4}$$

Now the matrix $[U]^{-1}$ is given in terms of the N-matrix entries U_m^{-1} by the same sort of expression as (2). Consequently, the matrix $[U]$ can be inverted in the following direct way: 1) express the eigenvalue λ_m, $m=1,\cdots,N$ as a linear function of the matrix entries U_m as in (3); 2) determine the scalar inverses $\lambda_m^{-1}=1/\lambda_m$; and 3) express the N-matrix entries U_m^{-1} as a linear function of λ_m^{-1}, $m=1,\cdots,N$ using the expression equivalent to (4), namely

$$U_m^{-1}=\frac{1}{N}\sum_{n=1}^{N} \lambda_n^{-1} e^{-j(m-1)(n-1)2\pi/N}. \tag{5}$$

If we define $m^*=m-1$ and $n^*=n-1$, then (3) and (5) become

$$\lambda_{m^*}=\sum_{n^*=0}^{N-1} U_{n^*} e^{j(2\pi m^* n^*/N)} \tag{6}$$

$$U_{m^*}^{-1}=\frac{1}{N}\sum_{n^*=0}^{N-1} (\lambda_{n^*})^{-1} e^{-j(2\pi m^* n^*/N)}. \tag{7}$$

But these expressions have the form of two discrete Fourier transforms (DFT)! In particular, the inversion of a symmetrical matrix with the form of $[U]$ in (2) requires two discrete Fourier transform operations and N divisions. The

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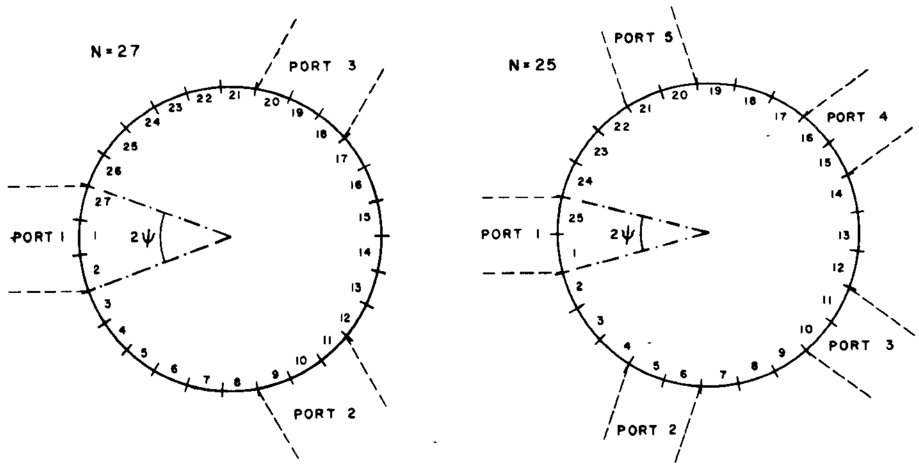

Fig. 1. Planar (a) 3- and (b) 5-port circuits with N-fold rotational symmetry, where N is the number of segments around the periphery.

evaluation of (1) including the matrix multiplication takes 3 N^2 multiplications, 3 $(N-1)^2$ additions, and N divisions.

III. Use of the FFT with the Integral Equation Method

Because two DFT's are required to invert the matrix U, it is possible to use a fast Fourier transform technique (FFT) instead to further substantially reduce the number of computations required. There are some restrictions on the FFT technique which may be employed. These will be discussed shortly. The FFT may also be used to reduce the number of operations needed to perform the matrix multiplication in (1). The method used is to first determine the eigenvalues of both the matrix $[U]$ and the matrix $[T]$ using two separate FFT's. The eigenvalues of the matrix Z in (1) can then be determined by dividing the N eigenvalues of $[T]$ by the N eigenvalues of $[U]$. The matrix entries of $[Z]$ are then recovered using a further FFT. This last operation is based on a DFT similar to that given in (7). Consequently, three FFT's are needed to perform the computations given in (1) for the case of N-fold rotational symmetry. All matrix operations can be eliminated.

Perhaps the most common FFT technique in use is the Cooley–Tukey procedure [4]. It assumes that N is a power of 2 so that $N=2^P$, where P is an integer. This procedure won't work here. From Fig. 1(a) and (b), it is apparent that N must contain 3 as a factor in the case of the symmetrical 3-port circulator of Fig. 1(a) and N must contain 5 as a factor in the case of the symmetrical 5-port of Fig. 1(b). Consequently, a different procedure must be used. In general, these procedures are based upon decomposing N into composite factors and carrying out Fourier transforms over the smaller number of terms in each of the composite factors. If N is the product of p factors, then

$$N=\prod_{i=1}^{p} r_i = r_1 r_2 \cdots r_p. \tag{8}$$

For the case of the symmetrical 3-port in Fig. 1(a), one of the factors r_i must be 3. In the case of the symmetrical 5-port in Fig. 1(b), one of the factors must be 5. The simplest assumption is to take them all to be 3 for the 3-port ($N=3^p$) and all to be 5 for the 5-port ($N=5^p$). The speed ratio improvement S of these FFT procedures to the DFT is given by

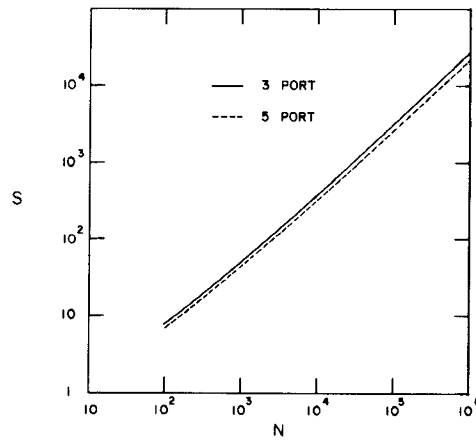

Fig. 2. Plots of the speed improvement factor S versus the number of points around the periphery N for both the 3-port and 5-port cases.

$$S=\frac{N^2}{N\sum_{i=1}^{p} r_i}=\frac{N}{\sum_{i=1}^{p} r_i}. \tag{9}$$

In the case of the N-fold symmetrical 3-port with all factors $r_i=3$, one has that $S=N/3p$. In the case of the N-fold symmetrical 5-port with all factors $r_i=5$, $S=N/5p$. For example, in the case of the 3-port with 729 points around the periphery corresponding to $p=6$, $S=729/3\times 6\simeq 40$. For the case of the 5-port with 625 points around the periphery corresponding to $p=4$, $S=625/5\times 4\simeq 31$. With roughly 1000 points around the periphery, it is possible to specify the coupling angle $2\cdot\psi$ of the striplines with the disk to about 1-percent accuracy. A plot of the speed improvement factor S versus the number of points around the periphery N is given in Fig. 2 for both the 3-port and 5-port cases.

IV. Application to Microstrip

Up until this time, the integral equation method has been applied to planar stripline circuits (Fig. 3(a)) with a dielectric constant ϵ and radius R on both sides of the

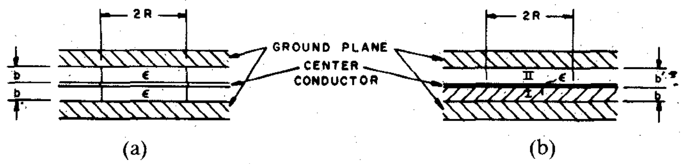

Fig. 3. (a) Stripline and (b) microstrip configurations for application of the FFT method.

center conductor. However, it can be applied as well to planar microstrip circuits (Fig. 3(b)). In this case, there is an air dielectric ($\epsilon = 1$) above the center conductor. In addition, the air gap distance to the ground plane b' may be taken to be different from the dielectric thickness b below the center conductor. Equation (1) is now replaced by two matrix equations. One of these applies to the air gap above the center conductor (region II) and the other applies to the dielectric medium below the center conductor (region I). That is

$$[Z] = [U]^{-1} \cdot [T], \quad [Z'] = [U']^{-1} \cdot [T'] \qquad (10)$$

where the $N \times N$ matrices $[U]$, $[T]$ are determined by physical parameters below the center conductor, and the $N \times N$ matrices $[U']$, $[T']$ are determined by physical parameters above the center conductor. The eigensusceptances determined from the matrices $[Z]$, $[Z']$ must then be summed to give the total eigensusceptances for the circuit. This is the case because the two regions above and below the center conductor are in shunt. These circuit eigensusceptances then determine the electrical performance. It should be clear that the computational time will be essentially doubled for the case of microstrip.

V. Conclusion

In summary, N-fold rotational symmetry is all that is required to apply the FFT method described in this paper. Reflection symmetry about the center conductor is not required.

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